数理解析研究所講究録
第 860 巻 1994 年 32-40

古田の不等式が成立する範囲について

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Abstract. Let $0 \leq p, q, r \in \mathbb{R}, p+2r \leq (1+2r)q$ and $1 \leq q$. Furuta ([1]) proved that if bounded linear operators $A, B \in B(H)$ on a Hilbert space H ($dim(H) \geq 2$) satisfy $O \leq B \leq A$, then $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$. In this paper, we prove that the range $p+2r \leq (1+2r)q$ and $1 \leq q$ is best possible with respect to Furuta's inequality, that is, if $(1+2r)q < p+2r$ or $0 < q < 1$, then there exixt $A, B \in B(\mathbb{R}^2)$ which satisfy $O \leq B \leq A$ but $B^{\frac{p+2r}{q}} \not\leq (B^r A^p B^r)^{\frac{1}{q}}$.

Let A, B be bounded linear operators on a Hilbert space H with $dim(H) \geq 2$. Furuta ([1]) proved a following interesting inequality.

Proposition 1 ([1]). *Let* $0 \leq p, q, r \in \mathbb{R}$ *and* $A, B \in B(H)$ *satisfy* $O \leq B \leq A$. *If*

$$p + 2r \leq (1+2r)q \quad \text{and} \quad 1 \leq q, \tag{1}$$

then

$$B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}. \tag{2}$$

This inequality (2) is an extension of Heinz's inequality ([2]) and many applications has been developped recently.

Proposition 2 ([2]). *Let* $A, B \in B(H)$ *satisfy* $O \leq B \leq A$. *If* $0 < \alpha < 1$, *then*

$$B^{\alpha} \leq A^{\alpha}.$$

Furuta caluculated many matrices, so the range (1) has been regarded as best possible. In this paper, we prove that the range (1) is indeed best possible with respect to Furuta's inequality, that is, if $(1+2r)q < p+2r$ or $0 < q < 1$, then there exixt $A, B \in B(\mathbb{R}^2)$ which satisfy $O \leq B \leq A$ but $B^{\frac{p+2r}{q}} \not\leq (B^r A^p B^r)^{\frac{1}{q}}$.

Since

$$\begin{aligned} &\{(p,q,r)\in \mathbb{R}_+^3 \mid O\le B\le A \Longrightarrow (2)\}\\ &=\{(p,q,r)\in \mathbb{R}_+^3 \mid O\le B\le A,\ A,\ B \text{ are invertible } \Longrightarrow (2)\}, \end{aligned}$$

we may assume A, B are invertible. Then $O \le B \le A$ is equivalent to $O \le A^{-1} \le B^{-1}$. Hence, by considering A^{-1}, B^{-1} instead of A, B, the inequality (2) becomes a following inequality

$$(3) \qquad (A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}}.$$

Hence

$$\begin{aligned} &\{(p,q,r)\in \mathbb{R}_+^3 \mid O\le B\le A \Longrightarrow (2)\}\\ &=\{(p,q,r)\in \mathbb{R}_+^3 \mid O\le B\le A, B \text{ is invertible } \Longrightarrow (3)\}\\ &=\{(p,q,r)\in \mathbb{R}_+^3 \mid O\le B\le A \Longrightarrow (3)\}. \end{aligned}$$

We prove the following theorem to show the best possibility of the range (1).

Theorem. *Let $0 < p, q, r \in \mathbb{R}$. If $(1 + 2r)q < p + 2r$ or $0 < q < 1$, then there exist $A, B \in B(\mathbb{R}^2)$ with $O \le B \le A$ which do not satisfy the inequality*

$$(3) \qquad (A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}}.$$

Proof. If A, B satisfy (3), then tA, tB $(0 < t)$ and U^*AU, U^*BU (U is unitary) satisfy (3). Hence it is no loss of generality that we assume $B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ $(0 < b < 1)$ and $A = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}$. Then a characteristic function of $A - B$ is

$$\Delta_{A-B}(t) = t^2 - (a_1 - 1 + a_2 - b)t + (a_1 - 1)(a_2 - b) - a_3^2.$$

Hence $O \le A - B$ implies

$$1 \le a_1,\ b \le a_2,\ a_3^2 \le (a_1 - 1)(a_2 - b).$$

Since

$$\Delta_A(t) = t^2 - (a_1 + a_2)t + a_1 a_2 - a_3^2,$$

eigen values of A are

$$a_1 + \varepsilon,\ a_2 - \varepsilon$$

where

$$2\varepsilon = -a_1 + a_2 + \sqrt{(a_1 - a_2)^2 + 4a_3^2} \ge 0.$$

Also since

$$\Delta_A(b) = b^2 - (a_1 + a_2)b + a_1 a_2 - a_3^2$$
$$\geq (a_2 - b)(2a_1 - 1 - b) \geq 0,$$

we have

$$b \leq a_2 - \varepsilon.$$

Rewrite $a_1 = a,\ a_2 = b + \varepsilon + \delta$. Then, summarizing above arguments, we will consider

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a-b-\delta)} \\ \sqrt{\varepsilon(a-b-\delta)} & b+\varepsilon+\delta \end{pmatrix} \tag{4}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \tag{5}$$

where

$$0 < b < 1 < a,\ 0 < \varepsilon,\ 0 < \delta,\ \varepsilon(1-b) \leq \delta(a - 1 + \varepsilon). \tag{6}$$

Since $O \leq B \leq A$ is obvious, we must prove that A, B do not satisfy the inequality (3) for some $a, b, \varepsilon, \delta$. We will define δ as a function of ε, and prove that A, B do not satisfy the inequality (3) by letting $\varepsilon \to +0$.

First we prove the case that $(1+2r)q < p + 2r$. Let a, b be constants (independent of ε and δ),

$$\gamma = a - b + \varepsilon - \delta$$

and

$$U = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\sqrt{a-b-\delta} \end{pmatrix}.$$

Then U is unitary and

$$U^*AU = \begin{pmatrix} a+\varepsilon & 0 \\ 0 & b+\delta \end{pmatrix}.$$

Then, by (3),

$$(U^*A^rUU^*B^pUU^*A^rU)^{\frac{1}{q}} \leq U^*A^{\frac{p+2r}{q}}U,$$

hence

$$\gamma^{-\frac{1}{q}} \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}^{\frac{1}{q}} \leq \begin{pmatrix} (a+\varepsilon)^{\frac{p+2r}{q}} & 0 \\ 0 & (b+\delta)^{\frac{p+2r}{q}} \end{pmatrix} \tag{7}$$

where

$$A_1 = (a+\varepsilon)^{2r}(a - b - \delta + \varepsilon b^p),$$
$$A_2 = (b+\delta)^{2r}(\varepsilon + b^p(a-b-\delta)),$$
$$A_3 = (a+\varepsilon)^r(b+\delta)^r(1-b^p)\sqrt{\varepsilon(a-b-\delta)}.$$

Let

$$D = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}$$

and

$$V = \frac{1}{\sqrt{A_1 - A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{A_1 - A_2 + \varepsilon_1} & \sqrt{\varepsilon_1} \\ \sqrt{\varepsilon_1} & -\sqrt{A_1 - A_2 + \varepsilon_1} \end{pmatrix}$$

where

$$2\varepsilon_1 = -A_1 + A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}.$$

Then V is unitary and

$$V^* D V = \begin{pmatrix} A_1 + \varepsilon_1 & 0 \\ 0 & A_2 - \varepsilon_1 \end{pmatrix}.$$

Hence, by (7),

$$\gamma^{-\frac{1}{q}} \begin{pmatrix} (A_1 + \varepsilon_1)^{\frac{1}{q}} & 0 \\ 0 & (A_2 - \varepsilon_1)^{\frac{1}{q}} \end{pmatrix} \leq \frac{1}{A_1 - A_2 + 2\varepsilon_1} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix}$$

where

$$\begin{aligned} B_1 &= (a+\varepsilon)^{\frac{p+2r}{q}}(A_1 - A_2 + \varepsilon_1) + (b+\delta)^{\frac{p+2r}{q}}\varepsilon_1, \\ B_2 &= (a+\varepsilon)^{\frac{p+2r}{q}}\varepsilon_1 + (b+\delta)^{\frac{p+2r}{q}}(A_1 - A_2 + \varepsilon_1), \\ B_3 &= ((a+\varepsilon)^{\frac{p+2r}{q}} - (b+\delta)^{\frac{p+2r}{q}})\sqrt{\varepsilon_1(A_1 - A_2 + \varepsilon_1)}. \end{aligned}$$

Hence

$$\begin{aligned} 0 \leq & \left| \gamma^{\frac{1}{q}} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix} - (A_1 - A_2 + 2\varepsilon_1) \begin{pmatrix} (A_1 + \varepsilon_1)^{\frac{1}{q}} & 0 \\ 0 & (A_2 - \varepsilon_1)^{\frac{1}{q}} \end{pmatrix} \right| \\ &= (A_1 - A_2 + 2\varepsilon_1)\{(a+\varepsilon)^{\frac{p+2r}{q}}(b+\delta)^{\frac{p+2r}{q}}(A_1 - A_2 + \varepsilon_1 + \varepsilon_1)\gamma^{\frac{2}{q}} \\ &\quad - (a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}(A_1 - A_2 + \varepsilon_1)(A_2 - \varepsilon_1)^{\frac{1}{q}} - (b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}\varepsilon_1(A_2 - \varepsilon_1)^{\frac{1}{q}} \\ &\quad - (a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}\varepsilon_1(A_1 + \varepsilon_1)^{\frac{1}{q}} - (b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}(A_1 - A_2 + \varepsilon_1)(A_1 + \varepsilon_1)^{\frac{1}{q}} \\ &\quad + (A_1 - A_2 + \varepsilon_1 + \varepsilon_1)(A_1 + \varepsilon_1)^{\frac{1}{q}}(A_2 - \varepsilon_1)^{\frac{1}{q}}\} \\ &= (A_1 - A_2 + 2\varepsilon_1) \\ &\quad \{(A_1 - A_2 + \varepsilon_1)((a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}})((b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}) \\ &\quad + \varepsilon_1((a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}})((b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}})\}. \end{aligned}$$

Since $0 < A_1 - A_2 + 2\varepsilon_1$, we have a following key inequality

$$\begin{aligned} (8) \quad & \varepsilon_1((A_1 + \varepsilon_1)^{\frac{1}{q}} - (b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}})((a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}) \\ & \leq (A_1 - A_2 + \varepsilon_1)((a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_1 + \varepsilon_1)^{\frac{1}{q}})((b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_2 - \varepsilon_1)^{\frac{1}{q}}). \end{aligned}$$

Now we estimate each term of the inequality (8) as far as order of ε and δ. o implies $o(\varepsilon)$ or $o(\delta)$, i.e., $\frac{o}{\varepsilon}, \frac{o}{\delta} \to 0$ $(\varepsilon, \delta \to +0)$.

Then

$$A_1 = a^{2r}(a-b)\left(1 + \left(\frac{2r}{a} + \frac{b^p}{a-b}\right)\varepsilon + \frac{-1}{a-b}\,\delta + o\right),$$

$$A_2 = b^{p+2r}(a-b)\left(1 + \frac{1}{b^p(a-b)}\,\varepsilon + \left(\frac{2r}{b} - \frac{1}{a-b}\right)\delta + o\right),$$

$$A_3^2 = a^{2r}b^{2r}(a-b)(1-b^p)^2\varepsilon\left(1 + \frac{2r}{a}\,\varepsilon + \left(\frac{2r}{b} - \frac{1}{a-b}\right)\delta + o\right),$$

$$\begin{aligned}\varepsilon_1 &= \frac{1}{2}(A_1 - A_2)\left(-1 + \sqrt{1 + \frac{4A_3^2}{(A_1-A_2)^2}}\right)\\ &= \frac{a^{2r}b^{2r}(1-b^p)^2\varepsilon}{a^{2r} - b^{p+2r}}\left(1 + \frac{o}{\varepsilon}\right),\end{aligned}$$

$$(b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} = (a-b)^{\frac{1}{q}}b^{\frac{p+2r}{q}}\left(1 + \frac{1}{q(a-b)}\,\varepsilon + \frac{1}{q}\left(\frac{p+2r}{b} - \frac{1}{a-b}\right)\delta + o\right),$$

$$(A_2 - \varepsilon_1)^{\frac{1}{q}} = (a-b)^{\frac{1}{q}}b^{\frac{p+2r}{q}}\left(1 + \frac{2a^{2r} - a^{2r}b^p - b^{2r}}{q(a-b)(a^{2r} - b^{p+2r})}\,\varepsilon + \frac{1}{q}\left(\frac{2r}{b} - \frac{1}{a-b}\right)\delta + o\right),$$

$$(b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_2-\varepsilon_1)^{\frac{1}{q}} = (a-b)^{\frac{1}{q}}b^{\frac{p+2r}{q}}\varepsilon\left(\frac{-(1-b^p)(a^{2r}-b^{2r})}{q(a-b)(a^{2r}-b^{p+2r})} + \frac{p}{qb}\,\frac{\delta}{\varepsilon} + \frac{o}{\varepsilon}\right),$$

$$A_1 - A_2 + \varepsilon_1 = (a-b)(a^{2r} - b^{p+2r})\left(1 + \frac{o}{\varepsilon}\right),$$

$$(a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_2-\varepsilon_1)^{\frac{1}{q}} = (a-b)^{\frac{1}{q}}(a^{\frac{p+2r}{q}} - b^{\frac{p+2r}{q}})\left(1 + \frac{o}{\varepsilon}\right),$$

$$(A_1+\varepsilon_1)^{\frac{1}{q}} - (b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} = (a-b)^{\frac{1}{q}}(a^{\frac{2r}{q}} - b^{\frac{p+2r}{q}})\left(1 + \frac{o}{\varepsilon}\right)$$

and

$$(a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} - (A_1+\varepsilon_1)^{\frac{1}{q}} = (a-b)^{\frac{1}{q}}a^{\frac{2r}{q}}(a^{\frac{p}{q}} - 1)\left(1 + \frac{o}{\varepsilon}\right).$$

Then, by (8),

$$\begin{aligned}(9)\qquad & a^{2r}b^{2r}(1-b^p)^2(a^{\frac{p+2r}{q}} - b^{\frac{p+2r}{q}})(a^{\frac{2r}{q}} - b^{\frac{p+2r}{q}})\left(1 + \frac{o}{\varepsilon}\right) \le \\ & a^{\frac{2r}{q}}b^{\frac{p+2r}{q}}(a-b)(a^{2r} - b^{p+2r})(a^{\frac{p}{q}} - 1)\left(\frac{-(1-b^p)(a^{2r}-b^{2r})}{q(a-b)(a^{2r}-b^{p+2r})} + \frac{p}{qb}\frac{\delta}{\varepsilon} + \frac{o}{\varepsilon}\right).\end{aligned}$$

We remark that

$$\liminf_{\varepsilon,\delta\to+0} \frac{\delta}{\varepsilon} = \liminf_{\varepsilon,\delta\to+0} \frac{1-b}{a-1+\varepsilon} = \frac{1-b}{a-1},$$

and the minimum of the right term of inequality (9) in which $\varepsilon, \delta \to +0$ will be realized if $\frac{\delta}{\varepsilon} = \frac{1-b}{a-1}$.

Define

$$\delta = \frac{1-b}{a-1}\varepsilon.$$

Then, by letting $\varepsilon \to +0$, (9) becomes

$$q(a-1)(1-b^p)^2(a^{\frac{p+2r}{q}} - b^{\frac{p+2r}{q}})(a^{\frac{2r}{q}} - b^{\frac{2r}{q}}) \leq$$

$$a^{\frac{2r}{q}-2r} b^{\frac{p+2r}{q}-2r-1}(a^{2r} - b^{p+2r})(a^{\frac{p}{q}}-1)\{p(1-b)(a-b)(a^{2r}-b^{p+2r}) - b(a-1)(1-b^p)(a^{2r}-b^{2r})\}.$$

Since

$$0 < \frac{p+2r}{q} - 2r - 1,$$

by letting $b \to +0$, we have

$$0 < q(a-1)a^{\frac{p+2r}{q}} a^{\frac{2r}{q}} \leq 0.$$

That is a contradiction.

Next we prove the case that $0 < q < 1$. Let b be constant (independent of ε, δ). We remark that

$$a \geq \frac{\varepsilon}{\delta}\,(1-b) + 1 - \varepsilon.$$

Define a and $\delta = \delta(\varepsilon)$ as

$$a = \frac{\varepsilon}{\delta}\,(1-b) + 1 \to 0 \ (\varepsilon \to +0).$$

Hence $\delta = o(\varepsilon)$ $(\varepsilon \to +0)$. Moreover, to simplify the estimation of (8), we let

$$\frac{\delta}{\varepsilon^2}, \frac{\delta^{2r}}{\varepsilon^{1+2r}}, \frac{\delta^{\frac{p}{q}}}{\varepsilon^{1+\frac{p}{q}}} \to 0 \ (\varepsilon \to +0).$$

(For example $\delta = \min(\varepsilon^3, \varepsilon^{\frac{2+2r}{2r}}, \varepsilon^{\frac{p+2r}{p}})$.)

Now we estimate each term of the inequality (8) as far as order of δ. Then

$$A_1 = \left(\frac{\varepsilon}{\delta}\right)^{1+2r} (1-b)^{1+2r} \left(1 + \frac{2r(1+\varepsilon)+1-b+\varepsilon b^p}{1-b}\,\frac{\delta}{\varepsilon} + o(\delta)\right),$$

$$A_2 = \frac{\varepsilon}{\delta}\, b^{p+2r}(1-b)\left(1 + \frac{b^p - b^{p+1}+\varepsilon}{b^p(1-b)}\,\frac{\delta}{\varepsilon} + \frac{2r}{b}\,\delta + o(\delta)\right),$$

$$A_3^2 = \left(\frac{\varepsilon}{\delta}\right)^{1+2r} (1-b)^{1+2r} b^{2r}(1-b^p)^2\varepsilon \left(1 + \left(1 + \frac{2r(1+\varepsilon)}{1-b}\right)\frac{\delta}{\varepsilon} + \frac{2r}{b}\,\delta + o(\delta)\right),$$

$$\varepsilon_1 = b^{2r}(1-b^p)^2\varepsilon\left(1 + \frac{-\varepsilon b^p}{1-b}\,\frac{\delta}{\varepsilon} + \frac{b^{p+2r}}{(1-b)^{2r}}\left(\frac{\delta}{\varepsilon}\right)^{2r} + \frac{2r}{b}\,\delta + o(\delta)\right),$$

$$(b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{q}} b^{\frac{p+2r}{q}}(1-b)^{\frac{1}{q}}\left(1+\frac{1-b+\varepsilon}{q(1-b)}\frac{\delta}{\varepsilon}+\frac{p+2r}{qb}\delta+o(\delta)\right),$$

$$(A_2-\varepsilon_1)^{\frac{1}{q}} = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{q}} b^{\frac{p+2r}{q}}(1-b)^{\frac{1}{q}}\left(1+\frac{1-b+2\varepsilon-b^p\varepsilon}{q(1-b)}\frac{\delta}{\varepsilon}+\frac{2r}{qb}\delta+o(\delta)\right),$$

$$(b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}-(A_2-\varepsilon_1)^{\frac{1}{q}} = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{q}} \frac{(1-b)^{\frac{1}{q}-1}b^{\frac{p+2r}{q}-1}(p-pb-b+b^{p+1})}{q}\,\delta\left(1+\frac{o(\delta)}{\delta}\right),$$

$$A_1-A_2+\varepsilon_1 = \left(\frac{\varepsilon}{\delta}\right)^{1+2r}(1-b)^{1+2r}\left(1+\frac{o(\delta)}{\delta}\right),$$

$$(a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}-(A_2-\varepsilon_1)^{\frac{1}{q}} = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1+p+2r}{q}}(1-b)^{\frac{1+p+2r}{q}}\left(1+\frac{o(\delta)}{\delta}\right),$$

$$(A_1+\varepsilon_1)^{\frac{1}{q}}-(b+\delta)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}} = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1+2r}{q}}(1-b)^{\frac{1+2r}{q}}\left(1+\frac{o(\delta)}{\delta}\right)$$

and

$$(a+\varepsilon)^{\frac{p+2r}{q}}\gamma^{\frac{1}{q}}-(A_1+\varepsilon_1)^{\frac{1}{q}} = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1+p+2r}{q}}(1-b)^{\frac{1+p+2r}{q}}\left(1+\frac{o(\delta)}{\delta}\right).$$

Then, by (8),

$$qb^{1+2r-\frac{p+2r}{q}}(1-b^p)^2(1-b)^{\frac{2r(1-q)}{q}}\left(1+\frac{o(\delta)}{\delta}\right) \le \left(\frac{\delta}{\varepsilon}\right)^{\frac{2r(1-q)}{q}}(p-pb-b+b^{p+1}).$$

Hence, by letting $\varepsilon \to +0$,

$$0 < qb^{1+2r-\frac{p+2r}{q}}(1-b^p)^2(1-b)^{\frac{2r(1-q)}{q}} \le 0.$$

That is a contradiction. **q.e.d.**

Remark. This Theorem shows that the range (1) is best possible with respect to Furuta's inequality if $\dim(H) \ge 2$.

Added in proof. There are more simple examples $A, B \in B(\mathbf{C}^2)$ in case of $(1+2r)q < p+2r$. To explain the examples we need following lemma.

Lemma. *Let* $a,\ b,\ d,\ \theta \in \mathbf{R}$ *satisfy* $0 < a+b,\ ab = d^2$ *and*

$$S = \begin{pmatrix} a & de^{-i\theta} \\ de^{i\theta} & b \end{pmatrix}.$$

Then

$$S^p = (a+b)^{p-1}S \quad \text{for} \quad 0 < p.$$

proof. Let

$$U = \frac{1}{\sqrt{b^2+d^2}}\begin{pmatrix} de^{-i\theta} & b \\ b & -de^{i\theta} \end{pmatrix}.$$

Then U is unitary and

$$U^*SU = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} S^p &= U(U^*SU)^p U^* \\ &= U \begin{pmatrix} (a+b)^p & 0 \\ 0 & 0 \end{pmatrix} U^* \\ &= (a+b)^{p-1} S. \end{aligned}$$

q.e.d.

Now we explain simple examples A, B. Let $0 < c < 1$, $\theta \in \mathbf{R}$,

$$A = \begin{pmatrix} 2 & 2\sqrt{c(1-c)}e^{i\theta} \\ 2\sqrt{c(1-c)}e^{-i\theta} & 4c \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $A - B$ is an Hermitian matrix and its characteristic function is

$$\Delta_{A-B}(t) = t^2 - (1+4c)t + 4c^2.$$

Hence $O \le B \le A$.

We prove A, B does not satisfy (3). Assume contrary A, B satisfy (3). Let

$$V = \begin{pmatrix} \sqrt{1-c}e^{i\theta} & \sqrt{c} \\ \sqrt{c} & -\sqrt{1-c}e^{-i\theta} \end{pmatrix}.$$

Then V is unitary and

$$V^*AV = \begin{pmatrix} 2+2c & 0 \\ 0 & 2c \end{pmatrix}.$$

By (3),

$$((V^*AV)^r V^* B^p V (V^*AV)^r)^{\frac{1}{q}} \le (V^*AV)^{\frac{p+2r}{q}}.$$

Hence, by Lemma,

$$\begin{aligned} &\begin{pmatrix} (2+2c)^{2r}(1-c) & -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{-i\theta} \\ -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{i\theta} & (2c)^{2r}c \end{pmatrix}^{\frac{1}{q}} \\ &= \delta \begin{pmatrix} (2+2c)^{2r}(1-c) & -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{-i\theta} \\ -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{i\theta} & (2c)^{2r}c \end{pmatrix} \\ &\le \begin{pmatrix} (2+2c)^{\frac{p+2r}{q}} & 0 \\ 0 & (2c)^{\frac{p+2r}{q}} \end{pmatrix} \end{aligned}$$

where

$$\delta = ((2+2c)^{2r}(1-c) + (2c)^{2r}c)^{\frac{1}{q}-1}.$$

Hence

$$0 \le \begin{pmatrix} (2+2c)^{\frac{p+2r}{q}} - \delta(2+2c)^{2r}(1-c) & \delta(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{-i\theta} \\ \delta(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{i\theta} & (2c)^{\frac{p+2r}{q}} - \delta(2c)^{2r}c \end{pmatrix}.$$

By taking a determinant of right matrix,

$$\begin{aligned} 0 \le &((2+2c)^{\frac{p+2r}{q}} - \delta(2+2c)^{2r}(1-c))((2c)^{\frac{p+2r}{q}} - \delta(2c)^{2r}c) \\ &- \delta^2(2+2c)^{2r}(2c)^{2r}c(1-c). \end{aligned}$$

Hence

$$\begin{aligned} &\delta(2+2c)^{\frac{p+2r}{q}}(2c)^{2r}c + \delta(2+2c)^{2r}(2c)^{\frac{p+2r}{q}}(1-c) \\ &\le (2+2c)^{\frac{p+2r}{q}}(2c)^{\frac{p+2r}{q}}, \end{aligned}$$

and

$$\begin{aligned} &\delta(2+2c)^{\frac{p+2r}{q}}2^{2r} + \delta(2+2c)^{2r}2^{\frac{p+2r}{q}}c^{\frac{p+2r}{q}-2r-1}(1-c) \\ &\le (2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}}c^{\frac{p+2r}{q}-2r-1}. \end{aligned} \tag{4}$$

Since

$$0 < \frac{p+2r}{q} - 2r - 1,$$

by letting $c \to +0$, we have

$$0 < (2^{2r})^{\frac{1}{q}-1}2^{\frac{p+2r}{q}}2^{2r} \le 0.$$

That is a contradiction.

REFERENCES

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